MATH 2028 Honours Advanced Calculus II
2021-22 Term 1
Problem Set 6

due on Nov 1, 2021 (Monday) at 11:59PM

Instructions: You are allowed to discuss with your classmates or seek help from the TAs but you are required to write/type up your own solutions. You can either type up your assignment or scan a copy of your written assignment into ONE PDF file and submit through Gradescope on/before the due date. Please remember to write down your name and student ID. **No late homework will be accepted.**

Problems to hand in

1. Calculate the line integral $\int_C F \cdot d\vec{r}$ where
   (a) $F(x,y) = (xy^3, 0)$ and C is the unit circle $x^2 + y^2 = 1$ oriented counterclockwise;
   (b) $F(x,y,z) = (y^2, z, -3xy)$ where C is the line segment from $(1,0,1)$ to $(2,3,-1)$.
2. Let C be the curve of intersection of the upper hemisphere $x^2+y^2+z^2 = 4$, $z \geq 0$ and the cylinder $x^2 + y^2 = 2x$, oriented counterclockwise as viewed from high above the xy-plane. Evaluate the line integral $\int_C F \cdot d\vec{r}$ where $F(x,y,z) = (y,z,x)$.
3. Evaluate the line integral $\int_C F \cdot d\vec{r}$ where $F : \mathbb{R}^2 \setminus \{(0,0)\} \to \mathbb{R}^2$ is the vector field
$$F(x,y) = \left(\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2} \right)$$
   and C is an arbitrary path from $(1,1)$ to $(2,2)$ not passing through the origin.
4. Determine which of the following vector field F is conservative on $\mathbb{R}^n$. For whose that are conservative, find a potential function f for it. For those that are not conservative, find a closed curve such that $\oint_C F \cdot d\vec{r} \neq 0$.
   (a) $F(x,y) = (y^2, x^2)$;
   (b) $F(x,y,z) = (y^2 z, 2xyz + \sin z, xy^2 + y\cos z)$.

Suggested Exercises

1. Calculate the line integral $\int_C F \cdot d\vec{r}$ where $F(x,y) = (y,x)$ and C is the following parametrized curve:
   (a) $\gamma(t) = (t,t)$, $0 \leq t \leq 1$;
   (b) $\gamma(t) = (t,t^2)$, $0 \leq t \leq 1$;
   (c) $\gamma(t) = (1-t, 1-t)$, $0 \leq t \leq 1$;
   (d) $\gamma(t) = (\cos^2 t, 1 - \sin^2 t)$, $0 \leq t \leq \frac{\pi}{2}$;
   (e) $\gamma(t) = (\sin 2t, 1 - \cos 2t)$, $0 \leq t \leq \frac{\pi}{4}$;
   (f) $\gamma(t) = (\cos t, 1 - \sin t)$, $0 \leq t \leq \frac{\pi}{2}$.

2. Repeat the exercise above with the vector field $F(x, y) = (y^2, x)$.

3. Calculate the line integral $\int_C F \cdot d\vec{r}$ where

   (a) $F(x, y, z) = (z, x, y)$ and C is the line segment from $(0, 1, 2)$ to $(1, -1, 3)$.

   (b) $F(x, y, z) = (y, 0, 0)$ where C is the intersection of the unit sphere $x^2 + y^2 + z^2 = 1$ and the plane $x + y + z = 0$, oriented counterclockwise as viewed from high above the xy-plane.

4. Determine which of the following vector field F is conservative on $\mathbb{R}^n$. For whose that are conservative, find a potential function f for it. For those that are not conservative, find a closed curve such that $\oint_C F \cdot d\vec{r} \neq 0$.

   (a) $F(x, y) = (x + y, x + y)$;

   (b) $F(x, y) = (e^x + 2xy, x^2 + y^2)$;

   (c) $F(x, y, z) = (x^2 + y + z, x + y^2 + z, x + y + z^2)$.

5. Calculate $\int_C F \cdot d\vec{r}$ where $F : \mathbb{R}^3 \to \mathbb{R}^3$ is the vector field

$$F(x, y, z) = \left(3x + y^2 + 2xz, 2xy + ze^{yz} + y, x^2 + ye^{yz} + ze^{z^2}\right)$$

   and C is the parametrized curve $\gamma : [0, 1] \to \mathbb{R}^3$ given by

$$\gamma(t) = \left(e^{t^7 \cos(2\pi t^{21})}, t^{17} + 4t^3 - 1, t^4 + (t - t^2)e^{\sin t}\right).$$

Challenging Exercises

1. Suppose $F : \mathbb{R}^n \to \mathbb{R}^n$ is a vector field on $\mathbb{R}^n$ defined by

$$F(x_1, x_2, \cdots, x_n) = (f(r)x_1, f(r)x_2, \cdots, f(r)x_n)$$

   where $f : \mathbb{R} \to \mathbb{R}$ is a given function and $r := \left(\sum_{i=1}^n x_i^2\right)^{\frac{1}{2}}$.

   (a) Suppose f is differentiable everywhere. Prove that for all $i, j = 1, \cdots, n$

$$\frac{\partial F_i}{\partial x_j} = \frac{\partial F_j}{\partial x_i}$$

   on $\mathbb{R}^n \setminus \{\vec{0}\}$ where F_k is the k-th component function of the vector field F.

   (b) Suppose f is continuous everywhere. Prove that F is a conservative vector field on $\mathbb{R}^n$.